Higher Order Poles in the S Matrix*

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A method is developed for studying the possibility of a coincidence of more than one complex pole of the S matrix so as to produce a higher order pole. It is shown thereby that complex higher order poles may be consistent with generalized unitarity, although a real higher order pole is not consistent with physical unitarity. Also discussed is the relevance of higher order poles, and of a group of simple poles, to the Wigner time-delay formula.

1. INTRODUCTION

In a recent paper[1] we remarked, in passing, that the Wigner time delay[2] for scattering can be defined usefully only under certain conditions (these are discussed more fully in Sec. 3 of this present paper), which can be satisfied at an energy near a group of resonances. Similar considerations have since appeared in a paper by Goldberger and Watson,[3] except that, instead of a group of resonances, they deal with a single higher order pole. We shall see in Sec. 3 that, in their effect on time delay in scattering, these situations may be qualitatively similar. This is because both can lead to rapid phase changes of the amplitude as the energy varies through the band defined by the wave packet of the incident particles. However, we shall see also that the experimental interpretations of the two situations may differ.

It is well known[4–6] that under quite general conditions the stable particle poles of the S matrix are simple; the two key properties that rule out real multiple poles are unitarity and the asymptotic condition. However, it seems that nothing is known that rules out the possibility of complex multiple poles and it is the object of this paper to consider a way in which they might occur.

It is believed[7] that, so far as analytic properties are concerned, complex poles produce effects very similar to those of real poles. In particular, they combine with each other or with stable particle poles to produce branch points whose corresponding discontinuities are given by expressions exactly similar to those for the stable particles. A simple way[8] to derive this property is to assume that somewhere in the theory there is a parameter g, for example a coupling constant, which can be varied in such a way that any given unstable particle becomes stable[9] and that everything changes analytically in the process. Although we cannot in a general theory justify such an assumption, we will make it here. It is fairly obvious that it is true in potential theory.

An immediate consequence of the assumption is that multiple poles only occur for discrete values of g. For if they occurred for a continuum of values, they would have to occur for all values, because of the analyticity assumption. But we have already said that multiple poles are excluded for those values of g for which they represent stable particles.

In the next section, therefore, we examine how poles that are normally simple might, for certain values of g, come together to form a multiple pole (for simplicity, we confine the discussion there to that of a double pole). Notice, however, that when two simple poles come together they do not necessarily form a double pole; they might equally well form a single simple pole. We have no general argument for predicting which does happen, assuming that a coincidence ever does happen. But we can remark on two particular examples. The first is that of potential theory, where[4] for potentials that have reasonable behavior

$$S_l(k)=e^{-2ikR}\prod_{n=1}^{\infty}\left(\frac{k-a_n-ib_n}{k-a_n+ib_n}\right)\left(\frac{k+a_n-ib_n}{k+a_n+ib_n}\right)\times\prod_{m=1}^{N}\left(\frac{k-ic_m}{k+ic_m}\right). \quad (1)$$

Here $a_n>0$, $b_n>0$, but c_m can have either sign. If by variation of one or more parameters g in the potential we can make two resonance poles coincide, the S matrix will then have a double pole because it is clear that there cannot also be a coincident zero. (If stable poles coincide, there must be a coincident zero, so that the resulting pole is simple.[4]) The other situation we can discuss, and it actually includes the first, is when the two poles under discussion occur in an elastic scattering amplitude on that unphysical sheet of the amplitude reached from the physical sheet by passing through the

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[9] If g is a coupling constant, we require that it be physical (real) at its initial and final values in this variation.

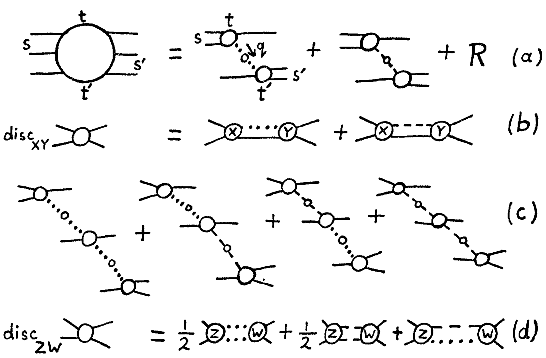

FIG. 1. (a) Two distinct simple poles of the six-point function that combine to give a double pole. (b) The discontinuity across the two cuts which combine to become the complex mM_0 cut. (c) Poles of the eight-point function that combine to become a product of two poles. (d) Discontinuities that combine to become the $2M_0$ discontinuity. Solid lines represent m particles, dotted lines M_1 particles, and dashed lines M_2 particles.

elastic cut. The discontinuity relation

$$a_l^{(2)}(s)-a_l^{(1)}(s)=i\rho(s)a_l^{(2)}(s)a_l^{(1)}(s)\,, \tag{2}$$

where $\rho(s)$ is a phase space factor and the superscripts 1, 2 denote the values of the amplitude, respectively, on the physical sheet and on the unphysical sheet described above, gives

$$a_l^{(2)}(s)=a_l^{(1)}(s)/[1-\rho(s)a_l^{(1)}(s)]. \tag{3}$$

From this we see that two coincident zeros of the denominator cannot fail to lead to a double pole of $a^{(2)}$, except on the real axis, because $a^{(1)}$ has no complex poles that could produce a cancellation.

The coincidence of poles can be examined qualitatively in potential theory using Regge surfaces. Let the Regge surfaces, in complex l, complex k space, corresponding to the poles of $S(l,k)$, be given by the equations

$$l=f_1(k,g)\,;\quad l=f_2(k,g)\,,$$

where again g denotes the coupling constant, or some set of parameters, in the potential. These surfaces in general will meet, for given g, in a discrete set of points, at which we obtain two coincident poles. As g is varied through real values these points trace out a one-dimensional curve, but for general variations of the potential they will trace out a four-dimensional volume in l, k space. This volume cannot include integer l and k in $\mathrm{Im}k>0$, but it has not been shown to exclude integer l and k complex in the lower half-plane. It is the latter possibility which concerns us in the next section.

2. DOUBLE POLES

Consider a theory that contains a stable particle of mass m and two particles whose masses M_1 and M_2 are analytic functions of our parameter g. As we have explained in the previous section, we assume that there exist ranges of values of g (not necessarily overlapping) such that in these ranges, the masses M_1, M_2, respectively, represent stable particles.

When M_1 is stable, the work of Olive[6] (and also familiar perturbation theory) shows that the six-point function $m+m+m\to m+m+m$ contains a simple pole at $q^2=M_1^2$, as depicted by the first term on the right-hand side of the symbolic equation drawn in Fig. 1(a). In Fig. 1, solid lines represent m particles, dotted lines M_1 particles, and dashed lines M_2 particles. Explicitly, the pole term is

$$A_1(s,t)A_1(s',t')/(q^2-M_1^2)\,, \tag{4}$$

where A_1 represents the scattering amplitude $m+m\to m+M_1$ and also, because of the symmetry of the S matrix, the amplitude $m+M_1\to m+m$. The variables s, s' are partial energies of the six-point function, while t, t' are momentum transfers, as marked in the figure. The four-momentum carried by the M_1 particle is called q.

When M_2 is stable there must be a corresponding contribution

$$A_2(s,t)A_2(s',t')/(q^2-M_2^2)\,, \tag{5}$$

depicted by the second term on the right-hand side of Fig. 1(a). Thus, our assumption about analyticity as g is varied requires that for all values of M_1, M_2, real or complex, the structure of the six-point function is

$$(4)+(5)+R\,,$$

where R is regular both at $q^2=M_1^2$ and at $q^2=M_2^2$.

Suppose now that g can be varied in such a way that $M_1\to M_2$. If A_1 and A_2 remain finite, the result is a single simple pole. If, however, we have the expansions

$$\begin{aligned}\sqrt{2}(M_i^2-M^2)^{1/2}A_i(s,t)&\\ =\alpha(s,t)+(M_i^2-M^2)\alpha'(s,t)+\cdots&\qquad i=1,2,\end{aligned} \tag{6}$$

where $M^2=\frac{1}{2}(M_1^2+M_2^2)$, then as $M_1\to M_2=M_0$

$$\begin{aligned}&A_1\to\alpha/(M_1^2-M_2^2)^{1/2}\,,\\ &A_2\to -iA_1.\end{aligned} \tag{7}$$

In this case we find that in this limit the six-point function has the double pole

$$\alpha(s,t)\alpha(s',t')/(q^2-M_0^2)^2 \tag{8}$$

for all values of s, t, s', t'. Superimposed on this will be the simple pole

$$\frac{\alpha(s,t)\alpha'(s',t')+\alpha'(s,t)\alpha(s',t')}{q^2-M_0^2}. \tag{9}$$

Of course (8) and (9) together are exactly similar to what would have been obtained by starting with a theory with only a simple pole at M_0^2 and then differentiating with respect to M_0^2. As remarked earlier, M_0 must be complex.

The discontinuity relations for branch cuts associated with double-pole thresholds are more complicated than in the simple-pole case. For example, consider the discontinuity of the $m+m\to m+m$ amplitude across the two cuts attached to the $(m+M_1)$, $(m+M_2)$ thresholds before the latter coincide to become the $(m+M_0)$ threshold. This discontinuity is drawn in Fig. 1(b). Here X, Y denote the limit to be taken in the total energy variable s; Y is reached from X by going round both the branch points. Explicitly, the right-hand side of the equation in Fig. 1(b) is

$$\rho_{mM_1}(X)\int A_1(X)A_1(Y)d\Gamma \\ +\rho_{mM_2}(X)\int A_2(X)A_2(Y)d\Gamma, \quad (10)$$

the integration being over the (real) angles Γ in the (complex) phase spaces of the intermediate states and the ρ being the appropriate phase space functions as in (2). When (6) is inserted in (10) the terms in $(M_1^2-M_2^2)^{-1}$ cancel, so that as $M_1\to M_2$ the discontinuity becomes

$$\int[(\alpha(X)\alpha'(Y)+\alpha'(X)\alpha(Y))\rho_{mM_0} \\ +\alpha(X)\alpha(Y)\rho']d\Gamma, \quad (11)$$

where ρ' denotes $\partial\rho_{mM_0}/\partial(M_0^2)$. Notice that if M_1 and M_2 were real, the cancellation could not occur, since $A_i(Y)$ would then be the complex conjugate of $A_i(X)$ and (7) cannot hold. This is in accord with previous proofs[4–6] that real double poles do not occur.

In order to examine the discontinuities across cuts attached to branch points associated with more than one M_0 particle, we need to know the structure of amplitudes with more than external M_i line. We assert that, as $M_1\to M_2$, the amplitude with r M_1 lines and s M_2 lines must tend to a finite function divided by

$$(M_1^2-M^2)^{r/2}(M_2^2-M^2)^{s/2}.$$

Because of crossing, it is immaterial whether the M_i lines are incoming or outgoing or a mixture.

We discuss this in detail for the case $r+s=2$. For the amplitude A_{ij} for $m+m\to M_i+M_j$ (or any amplitude obtained from this by crossing), we require the expansion

$$\begin{aligned}\tfrac{1}{2}(M_i^2-M^2)^{1/2}&(M_j^2-M^2)^{1/2}A_{ij}\\ &=\beta(s,t)+(M_i^2-M^2)\beta^{(10)}+(M_j^2-M^2)\beta^{(01)}\\ &\quad+\tfrac{1}{2}[(M_i^2-M^2)^2\beta^{(20)}+2(M_i^2-M^2)(M_j^2-M^2)\beta^{(11)}\\ &\qquad+(M_j^2-M^2)^2\beta^{(02)}]+\cdots, \quad (12)\end{aligned}$$

where, as $M_1\to M_2$,

$$\beta^{(01)}\to\beta^{(10)}=\beta', \\ \beta^{(20)}\to\beta^{(11)}\to\beta^{(02)}=\beta''. \quad (13)$$

We obtain this structure from the requirement that the sum of the four twofold pole terms in Fig. 1(c) should tend, as $M_1\to M_2$, to a product of two double poles (together with less singular terms). It is also consistent with the three discontinuity equations for the (2m) cut:

$$\mathrm{disc}A_{ij}=\rho_{mm}\int A_iA_jd\Gamma, \quad (14)$$

the structure of the right-hand side of this equation having already been fixed by (6).

We can now find the discontinuity of the $m+m\to m+m$ amplitude across the $2M_0$ cut. We first find the discontinuity across the $2M_1$, $2M_2$, (M_1+M_2) cuts before these coincide. This is drawn in Fig. 1(d), where the boundary value W is reached from Z by going round all three branch points. The factors $\frac{1}{2}$ appear in these equations because one requires a factor $1/n!$ whenever the intermediate state contains n identical particles. Explicitly, the discontinuity reads

$$\tfrac{1}{2}\rho_{M_1M_1}\int A_{11}(Z)A_{11}(W)d\Gamma \\ +\tfrac{1}{2}\rho_{M_2M_2}\int A_{22}(Z)A_{22}(W)d\Gamma \\ +\rho_{M_1M_2}\int A_{12}(Z)A_{12}(W)d\Gamma. \quad (15)$$

If (12) is now inserted and use made of (13), the singular terms in (15) are found to cancel, leaving for the discontinuity across the $2M_0$ cut when $M_1\to M_2$:

$$\int[(\beta'(Z)\beta'(W)+\beta(Z)\beta''(W)+\beta''(Z)\beta(W))\rho_{M_0M_0} \\ +\beta(Z)\beta(W)\rho''+\beta'(Z)\beta(W)\rho' \\ +\beta(Z)\beta'(W)\rho']d\Gamma. \quad (16)$$

The discontinuity across branch cuts involving a double pole is the derivative with respect to the (complex) mass of the corresponding discontinuity involving a simple pole. However, this differentiation should be used with care as it applies only at the double-pole singularities. For example, the discontinuity formulas for the amplitudes across the $2m$ cut are the same as usual.

In concluding this section we wish to note that our discussion does not establish that higher order poles in the S matrix will occur. We have shown that the possibility of their occurrence is consistent with generalized discontinuity relations provided their location is complex. In particular we have shown how these discontinuity relations can be studied in the case of the coincidence of two complex simple poles with diverging residues that lead to a double pole.

3. LIFETIMES AND RESONANCE POLES

We have calculated[1] the extra flight time for a wave packet ψ due to interaction with a target and obtain as the average value:

$$\langle\tau\rangle=-\tfrac{1}{2}i\int dE\left[(S^*\psi)\frac{\partial}{\partial E}(S\psi)-(S\psi)\frac{\partial}{\partial E}(S^*\psi)\right]\Big/\int dE|S\psi|^2. \quad (17)$$

If the variation of $S'(E)/S(E)$ is small within the width α of the wave packet $\psi(E)$ this approximates to the Wigner[2] time delay:

$$\langle\tau\rangle\approx\tau_W=\mathrm{Re}[-i(\partial/\partial E)\ln S]. \quad (18)$$

The condition

$$S'(E)/S(E)\approx\text{constant}, \quad (19)$$

within the width α is the condition that $\langle\tau\rangle$ shall not change rapidly as the shape and position of the maximum of ψ are varied. If this is not fulfilled, $\langle\tau\rangle$ is of little physical interest anyway. Another condition that must be satisfied for $\langle\tau\rangle$, or τ_W, to be of physical interest is that it be large compared with the uncertainty principle error $1/\alpha$ that arises from the fact that the energy is known to within an accuracy α:

$$\tau_W\gg 1/\alpha. \quad (20)$$

If we integrate (19) we find that

$$S(E)\approx Ce^{iE\tau_W}, \quad (21)$$

within the width α, so that (20) requires S to undergo many phase changes of 2π within this width.

Such a behavior would not be obtained from a nearby simple complex pole, so we cannot use (18) to calculate the interaction time associated with the formation and decay of the single unstable particle represented by this pole. It can, however, be obtained if there is a nearby *group* of resonances, such as is obtained, for example, in the compound-nucleus model of resonance scattering.

To see how this works, consider for simplicity a group of equally spaced poles:

$$S=\prod_{\nu=-N}^{N}\frac{E-E_0-\nu\delta-i\beta}{E-E_0-\nu\delta+i\beta}. \quad (22)$$

Suppose the wave packet ψ is centered on E_0 and has width α rather less than $2N\delta$, so that the poles extend some way to either side of the wave packet. Then condition (19) requires

$$\delta\ll\beta,$$

so that the spacing of the poles must be much less than their distance from the real axis. If α is roughly $N\delta$, this is

$$\alpha/N\ll\beta. \quad (23)$$

We examine two ways in which (23) can be satisfied. The first is

$$\alpha\ll\beta. \quad (24)$$

Then (18) and (22) give

$$\tau_W\approx 4N/\beta,$$

so that (20) requires

$$\beta\ll N\alpha.$$

This case is qualitatively very similar to that of a single multiple pole[2] and would be experimentally indistinguishable from it.

The other situation to consider, since N has to be large, is that as well as (23), we have

$$\alpha\sim\beta \quad (25)$$

or even,

$$\alpha\gg\beta. \quad (26)$$

Then from

$$\tau_W=\sum_{\nu=-N}^{N}\frac{2\beta}{\nu^2\delta^2+\beta^2}, \quad (27)$$

we obtain

$$\tau_W>4N\beta/(N^2\delta^2+\beta^2),$$

and, using $N\delta\sim\alpha$, this is of order $N\beta/\alpha^2$ and so is guaranteed to be much greater than $1/\alpha$ by (23). So here we have a case in which the Wigner formula is meaningful but which does not behave qualitatively in the same way as a single multiple pole. The energy width α of the wave packet is large enough to straddle many resonances ($N\gg 1$), but the associated time uncertainty $1/\alpha$ is smaller than, or of the same magnitude as, the lifetime $1/\beta$ that might be attributed to any one of the resonances alone. We cannot in this case ask which of the separate resonances is producing the interaction and consequent time delay.

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